CONCEPTS OF COMPLEX FUNCTION AND LIMIT OF A COMPLEX FUNCTION

Ref: Complex Variables by James Ward Brown and Ruel V.Churchil

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Definition

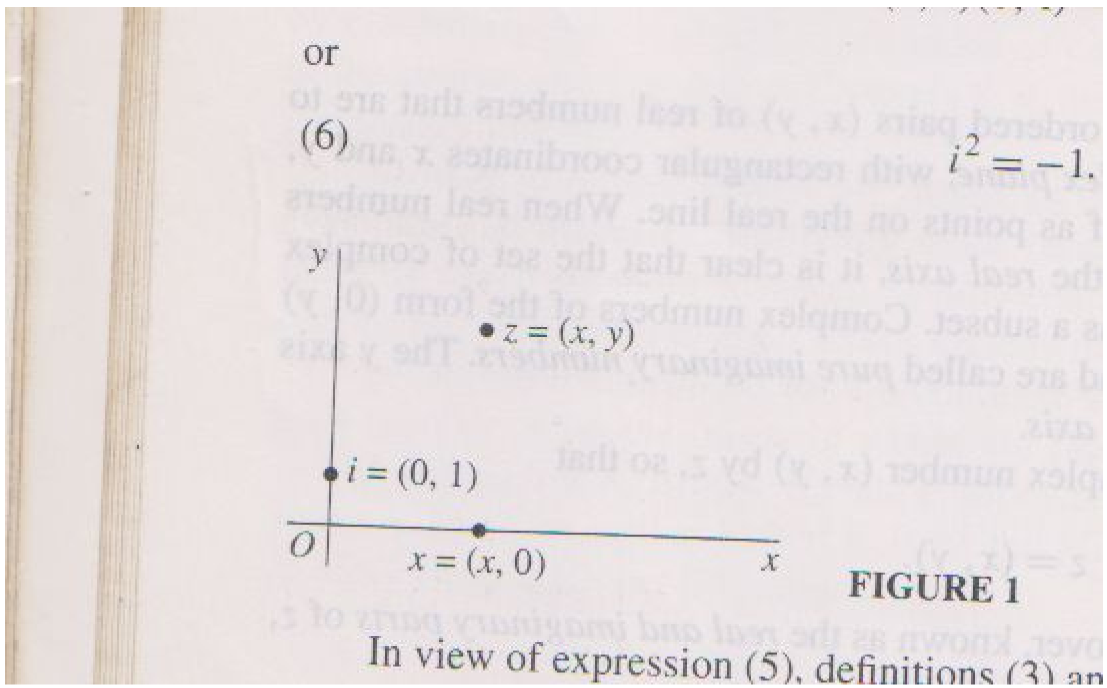
- Form 1. A number of the form $x + iy$, $x \in R$, $y \in R$ is called a complex number.
- Form 2. Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the complex plane with rectangular co-ordinates x and y.
- **Example**

  $1 + 2i = (1, 2)$

Diagrammatic representation

- z = (x, y) $\in$ C.

$\Rightarrow$ Re z = x,   Im z = y



Operations on complex numbers

- **Equality of Complex Numbers**

  Two complex numbers $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ are equal iff $x_1 = x_2$, $y_1 = y_2$.
- **Addition of Complex Numbers**

  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
- **Multiplication**

  $(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$
- $i = (0, 1)$ $i^2 = (0, 1)(0, 1) = (-1, 0)$

  $\therefore i^2 = -1$

Find the modulus of $-3 + 2i$

$|-3 + 2i| = \sqrt{4+9} = \sqrt{13}$

- $|z| = |x + iy| = \sqrt{x^2 + y^2}$
- **Equation** of circle with centre at origin and radius R is $x^2 + y^2 = R^2$, that is, $|z| = R$.
- **Circle** with centre z_0 radius R is $|z - z_0| = R$. Here z is a varying quantity.
- **What** is the centre and radius of the circle $|z - 1 + 3i| = 2$?

Radius = 2, Centre = (1, -3)

Results

Let x = x + iy = (x, y), then we have

- Re z = x, Im z = y
- $| z |^2 = (\text{Re } z)^2 + (\text{Im } z)^2$
- $\text{Re } z \leq | \text{Re } z | \leq | z |$
- $\text{Im } z \leq | \text{Im } z | \leq | z |$
- $| z_1 + z_2 | \leq | z_1 | + | z_2 |$
- $| z_1 - z_2 | \geq \big| | z_1 | - | z_2 | \big|$
- $| z_1 + z_2 | = | z_1 - (-z_2) | \geq \big| | z_1 | - | z_2 | \big|$

Complex Conjugates

- ***Form1:*** If $z = x + iy$ then the complex conjugate is $\bar{z} = x - iy$.

  Form2: If $z = (x, y)$ then the complex conjugate is $\bar{z} = (x, -y)$, which is the reflection in the real axis of the point (x, y).

Let $z = x + iy$, $z_1 = x_1 + iy_1$; $z_2 = x_2 + iy_2$,

Then,

(1) $\bar{\bar{z}} = z$

(2) $|\bar{z}| = |z| = \sqrt{x^2 + y^2}$

(3) $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$

(4) $\overline{z_1 \, z_2} = \overline{z}_1 \, \overline{z}_2$

(5) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2},\ z_2 \neq 0$

(6) $\operatorname{Re} z = \dfrac{z + \overline{z}}{2};\ \operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$

(7) $z\overline{z} = |z|^2 = |\overline{z}|^2$

Polar & Exponential form

- Let $z = x + iy$
- Let $x = r\cos\theta$, $y = r\sin\theta$
- Then $z = r(\cos\theta + i\sin\theta)$ is the **polar** form

and $z = re^{i\theta}$ is the **exponential** form.

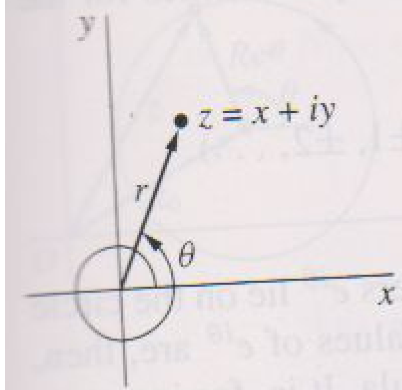


FIGURE 6

parametric representation of a circle &**Products and Quotients in Exponential form**

- The **parametric** representation of a circle $|\, z \,| = R$ is $z = \mathrm{R}e^{i\theta}$ $(0 \le \theta \le 2\pi)$
- The parametric representation of the circle

  $|\, z - z_0 | = R$ is $z = z_0 + \mathrm{R}e^{i\theta}$ $(0 \le \theta \le 2\pi)$
- **Products and Quotients in Exponential form**

  Suppose $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, then

  $z_1\, z_2 = r_1\, r_2\, re^{i(\theta_1+\theta_2)}$

  $$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1+\theta_2)}$$

- **De Moivere's Theorem**

  $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

  That is, $(e^{i\theta})^n = e^{in\theta}$,n $\in$ Z, the set of integers.

- **ε - neighbourhood**

  An ε-neighbourhood of a given point z_{-0} in the z-plane is $|z - z_0| < \varepsilon$ i.e., It consists of all points z lying inside but not on a circle centered at z_0 and with a specified positive radius ε.

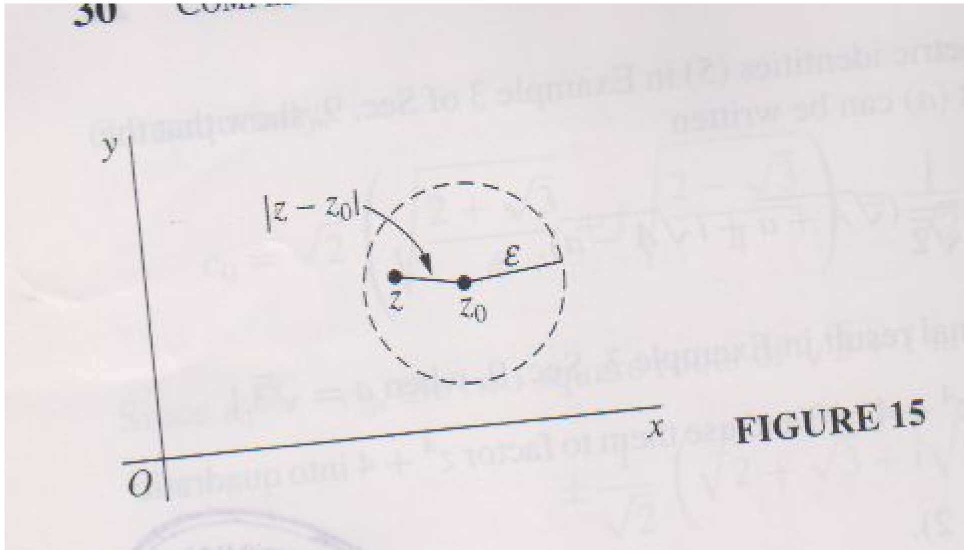


FIGURE 15

- **Deleted Neighbourhood**

  A deleted neighbourhood of z_0 in the z-plane is $0 < | z – z_0| < \varepsilon$.

  It consists of all points z in an ε-neighbourhood of z_0 except for the point z_0 itself.

- **Interior Point**

  A point z_{-0} is an interior point of a set S whenever there is some neighbourhood of z_{-0} that contains only points of S.

- **Exterior Point**

  A point z_{-0} is called an exterior point of S whenever there is some neighbourhood of z_0 that contains no points of S.

- **Boundary point**:

  z_0 is a boundary of S if z_0 is neither an interior point nor an exterior point of S. i.e., z_0 is a boundary point of S if there is some neighbourhood of z_0 that contains points both in S and not in S.

- **Example:** The circle $| z | = 1$ is the boundary of $| z | < 1$ and $| z | \leq 1$.

Functions of a complex variable

Let S be a set of complex numbers. A function f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the value of f at z and is denoted by f(z); i.e., $w = f(z)$.

Note

The set S is called the domain of the definition of f.

Example

If f is defined on the set $z \neq 0$ by means of the equation $w = \frac{1}{z}$, it may be referred to as the function $w = \frac{1}{z}$, or simple the function $\frac{1}{z}$.

Example. $f(z) = z^2$ is a function.

Result. Consider a function $w = f(z)$

$z = x + iy$; $w = u + iv$, where $u = u(x, y)$ and $v = v(x, y)$

i.e., u and v are real valued functions.

Example 1

$$w = \frac{1}{z} \Rightarrow u + iv = \frac{1}{x + iy}$$

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}.$$

Mappings

Consider the function w = f(z), where z = (x, y) and w = (u, v).

Properties of a real-valued function of a real variable are often exhibited by the graph of the function.

But for the function w = f(z), no such convenient graphical representation of the function f is available. Because each of the numbers, z and w is located in a plane rather than on a line.

However, we can display some information about the function w=f(z) by indicating pairs of corresponding points z = (x, y) and w = (u, v). In diagrammatic form by drawing the z and w planes separately.

When a function f is viewed of in this way it is often referred to as mapping or transformation.

Translation

The mapping w = z + 1 = (x + 1) + iy is called a translation of each point z one unit to the right.

Rotation

We know $i = e^{\frac{i\pi}{2}}$

$$\therefore\ w = iz = e^{\frac{i\pi}{2}} re^{i\theta} = re^{i\left(\theta+\frac{\pi}{2}\right)}$$

Here w = iz is called the rotation and it rotates the radius vector for each non-zero point z through a right angle about the origin in the counter clock wise direction.

Reflection

The mapping $w = \overline{z} = x - iy$ transforms each point z = x + iy into its reflection in the real axis.

Translation

The mapping w = z + 1 = (x + 1) + iy is called a translation of each point z one unit to the right.

Rotation

We know $i = e^{\frac{i\pi}{2}}$

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CHAPTER 8. MAPPING BY ELEMENTARY FUNCTIONS

83. Linear transformation

Expansion or contraction and a rotation

Describe w = AZ …………..(1)

Where A is a non zero complex constant and $z \neq 0$, we write A and z in the exponential form $A = ae^{i\alpha}$, $z = re^{i\theta}$.

Then $w = (ar)e^{i(\alpha+\theta)}$ ……………………..(2)

We see from equation (2) that transformation (1) expands or contracts the radius vector representing z by the factor $a=|A|$ and rotates it through an angle $\alpha=\arg A$ about the origin. The image of a given region is, therefore, geometrically similar to that region.

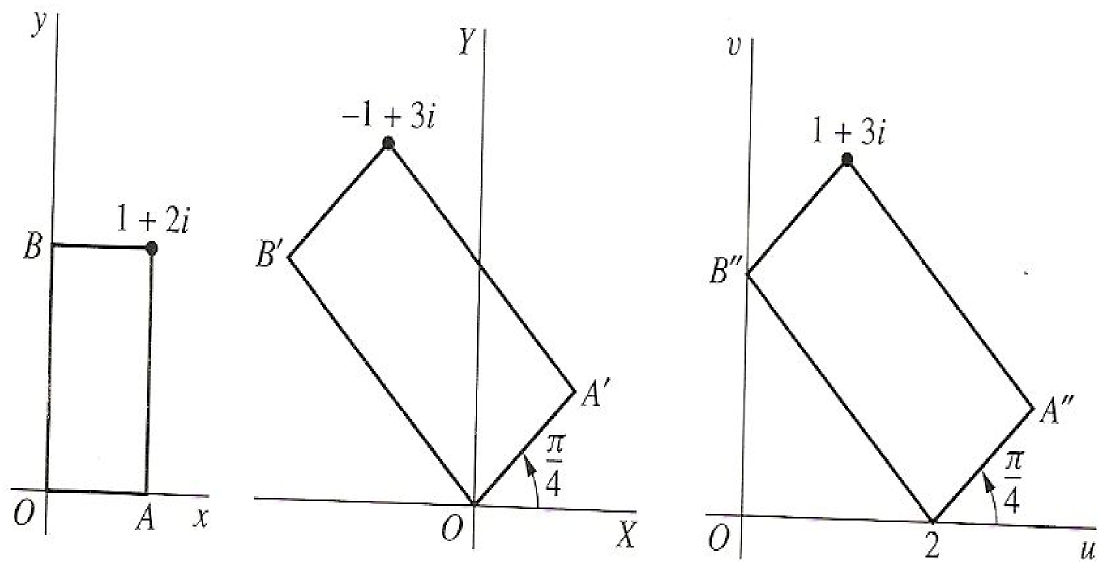

FIGURE 104
$w = (1 + i)z + 2.$

Translation: The mapping $w = z+B$………….(3) where B is any complex constant, is a translation by means of the vector representing B. Let $w= u+iv$, $z=x+iy$, $B= b_1+ib_2$. The image of any point (x, y) in the z plane is the point $(u, v) = (x+b_1, y+b_2)$ in the w plane. The image region is geometrically congruent to the original region.

Describe the linear transformation w = Az+B, $(A \neq 0)$**.**

It is an expansion or contraction and a rotation, followed by a translation: Consider the general (non constant) linear transformation

W= Az+B $(A \neq 0)$ …………….(5).

It is composition of $Z = Az (A \neq 0)$ and w= Z+B. It is an expansion or contraction and a rotation, followed by a translation.

Example

Give a geometric description of the transformation $w=(1+i)z+2$ (or) Find the image of the rectangular region with corner points (0,0), (1,0), (2,0), (1,2) in the z plane under the transformation $w=(1+i)z+2$ sketch the rectangle and the image.

Example

Give a geometric description of the transformation $w=(1+i)z+2$ (or) Find the image of the rectangular region with corner points (0,0), (1,0), (2,0), (1,2) in the z plane under the transformation $w=(1+i)z+2$ sketch the rectangle and the image.

Solution

$w=(1+i)z+2=Z+2$ where $Z=(1+i)z$. Let 1+i = a+ib.

Then $r=\sqrt{2}$ and $\theta=\tan^{-1}\frac{b}{a}=\tan^{-1}1=\frac{\pi}{4}$.

So 1+i $=\sqrt{2}$ exp $\left(i\frac{\pi}{4}\right)$.

So Z = (1+i) z is an expansion by the factor $\sqrt{2}$ and a rotation through the angle $\frac{\pi}{4}$ and w= Z+2 is a translation of two units to the right.

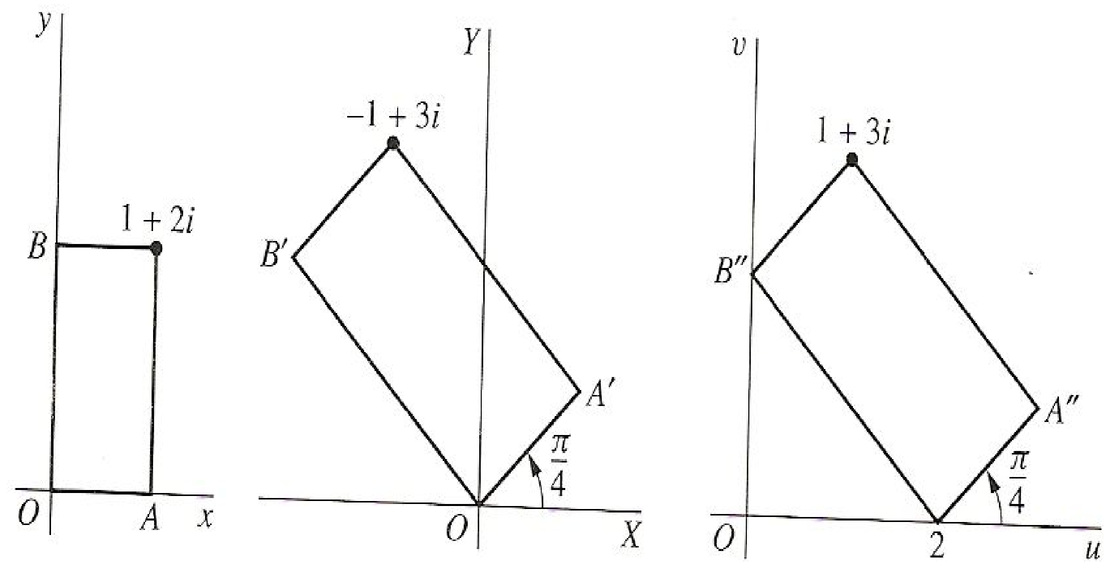

FIGURE 104
$w = (1 + i)z + 2.$

Limits

Consider the function w = f(z). The limit of f(z) as z approaches z_0 is a number w_0. ie., $\lim_{z \to z_0} f(z) = w_0$, <u>means</u> that the point w = f(z) can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it.

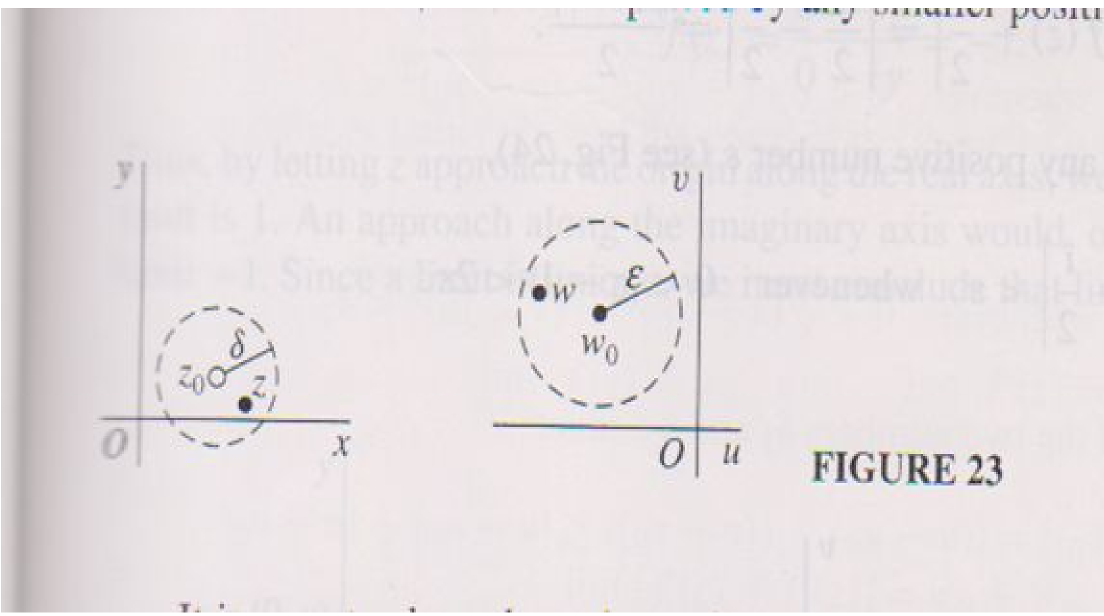

FIGURE 23

Definition Consider a function $w = f(z)$. We say $\lim_{z \to z_0} f(z) = w_0$ iff given $\varepsilon > 0$, there exists $\delta > 0$ such that $| f(z) - w_0| < \varepsilon$ whenever $0 < | z - z_0| < \delta$.

$$(0 < | z - z_0 | < \delta \Rightarrow | f(z) - w_0| < \varepsilon)$$

Theorem

When a limit of a function $f(z)$ exists at a point z_0 it is unique.

Proof:

Suppose that $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} f(z) = w_1$

Let $\varepsilon > 0$ be given (arbitrary).

Now $\lim\limits_{z \to z_0} f(z) = w_0 \Rightarrow$ there exists $\delta_0 > 0$ such that

$$0 < | z - z_0| < \delta_0 \Rightarrow | f(z) - w_0| < \varepsilon \qquad \ldots (1)$$

Now, $\lim\limits_{z \to z_0} f(z) = w_1 \Rightarrow$ there exists $\delta_1 > 0$ such that

$$0 < | z - z_0| < \delta_1 \Rightarrow | f(z) - w_1| < \varepsilon \qquad \ldots (2)$$

Let $\delta = \min\{\delta_0, \delta_1\}$.

Consider the deleted δ-neighbourhood $0 < |z - z_0| < \delta$.

Clearly if $0 < |z - z_0| < \delta$, then both (1) and (2) are true.

$$
\begin{aligned}
\text{Now, } |w_1 - w_0| &= |f(z) - w_0 + w_1 - f(z)| \\
&= |(f(z) - w_0) - (f(z) - w_1)| \\
&\leq |f(z) - w_0| + |f(z) - w_1|, \text{ whenever } 0 < |z - z_0| < \delta.
\end{aligned}
$$

$\therefore$ Given any $2\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z_1 - z_0| < \delta \Rightarrow |w_1 - w_0| < 2\varepsilon.$$

Since this is true for any $2\varepsilon > 0$, we get $|w_1 - w_0| = 0 \Rightarrow w_1 - w_0 = 0$.

$\Rightarrow w_1 = w_0. \quad \Rightarrow \lim\limits_{z \to z_0} f(z) = w_0 = w_1$

$\therefore$ If the limit of a function $f(z)$ exists at a point z_0, it is unique.